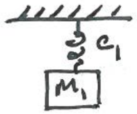
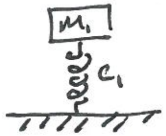
1-D Systems: Equilibrium & Stiffness Examples

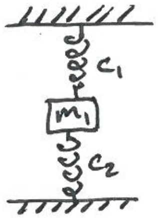
Write elongation and stiffness matrices for the following examples.

I) 

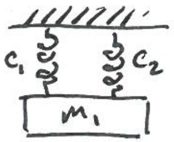
$$A = \begin{array}{cc} \text{mass 1} & \\ \begin{bmatrix} 1 \end{bmatrix} & \text{spring 1} \end{array} \qquad K = \begin{array}{cc} \text{mass 1} & \\ \begin{bmatrix} c_1 \end{bmatrix} & \text{mass 1} \end{array}$$

II) 

$$A = \begin{array}{cc} \text{mass 1} & \\ \begin{bmatrix} -1 \end{bmatrix} & \text{spring 1} \end{array} \qquad K = \begin{array}{cc} \text{mass 1} & \\ \begin{bmatrix} c_1 \end{bmatrix} & \text{mass 1} \end{array}$$

III) 

$$A = \begin{array}{cc} \text{mass 1} & \\ \begin{bmatrix} 1 \\ -1 \end{bmatrix} & \begin{matrix} \text{spring 1} \\ \text{spring 2} \end{matrix} \end{array} \qquad K = \begin{array}{cc} \text{mass 1} & \\ \begin{bmatrix} c_1 + c_2 \end{bmatrix} & \text{mass 1} \end{array}$$

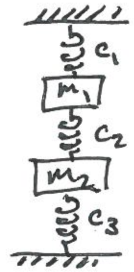
IV) 

$$A = \begin{array}{cc} \text{mass 1} & \\ \begin{bmatrix} 1 \\ 1 \end{bmatrix} & \begin{matrix} \text{spring 1} \\ \text{spring 2} \end{matrix} \end{array} \qquad K = \begin{array}{cc} \text{mass 1} & \\ \begin{bmatrix} c_1 + c_2 \end{bmatrix} & \text{mass 1} \end{array}$$

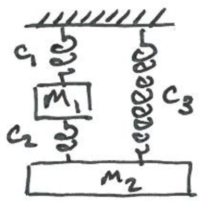
Note: Stiffness matrix, K, doesn't know which side of a mass springs connect to...

V) 

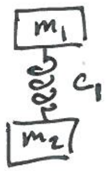
$$A = \begin{array}{cc} \begin{matrix} \text{mass 1} & \text{mass 2} \end{matrix} & \\ \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} & \begin{matrix} \text{spring 1} \\ \text{spring 2} \end{matrix} \end{array} \qquad K = \begin{array}{cc} \begin{matrix} \text{mass 1} & \text{mass 2} \end{matrix} & \\ \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 \end{bmatrix} & \begin{matrix} \text{mass 1} \\ \text{mass 2} \end{matrix} \end{array}$$

VI) 

$$A = \begin{array}{cc} \begin{matrix} \text{mass 1} & \text{mass 2} \end{matrix} & \\ \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & -1 \end{bmatrix} & \begin{matrix} \text{spring 1} \\ \text{spring 2} \\ \text{spring 3} \end{matrix} \end{array} \qquad K = \begin{array}{cc} \begin{matrix} \text{mass 1} & \text{mass 2} \end{matrix} & \\ \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} & \begin{matrix} \text{mass 1} \\ \text{mass 2} \end{matrix} \end{array}$$

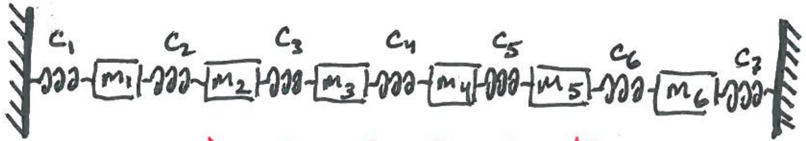
VII) 

$$A = \begin{array}{cc} \begin{matrix} \text{mass 1} & \text{mass 2} \end{matrix} & \\ \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 1 \end{bmatrix} & \begin{matrix} \text{spring 1} \\ \text{spring 2} \\ \text{spring 3} \end{matrix} \end{array} \qquad K = \begin{array}{cc} \begin{matrix} \text{mass 1} & \text{mass 2} \end{matrix} & \\ \begin{bmatrix} c_1 + c_2 & -c_2 \\ -c_2 & c_2 + c_3 \end{bmatrix} & \begin{matrix} \text{mass 1} \\ \text{mass 2} \end{matrix} \end{array}$$

VIII) 

$$A = \begin{array}{cc} \begin{matrix} \text{mass 1} & \text{mass 2} \end{matrix} & \\ \begin{bmatrix} 1 & -1 \end{bmatrix} & \text{spring 1} \end{array} \qquad K = \begin{array}{cc} \begin{matrix} \text{mass 1} & \text{mass 2} \end{matrix} & \\ \begin{bmatrix} c_1 & -c_1 \\ -c_1 & c_1 \end{bmatrix} & \begin{matrix} \text{mass 1} \\ \text{mass 2} \end{matrix} \end{array}$$

EX: Write the elongation and stiffness matrices for the "long line of springs with fixed-fixed ends":



$$A = \begin{array}{c c}
\begin{array}{cccccc} \text{mass 1} & \text{mass 2} & \text{mass 3} & \text{mass 4} & \text{mass 5} & \text{mass 6} \end{array} & \\
\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{bmatrix} & \begin{array}{l} \text{spring 1} \\ \text{spring 2} \\ \text{spring 3} \\ \text{spring 4} \\ \text{spring 5} \\ \text{spring 6} \\ \text{spring 7} \end{array}
\end{array}$$

$$K = \begin{array}{c c}
\begin{array}{cccccc} \text{mass 1} & \text{mass 2} & \text{mass 3} & \text{mass 4} & \text{mass 5} & \text{mass 6} \end{array} & \\
\begin{bmatrix} c_1+c_2 & -c_2 & 0 & 0 & 0 & 0 \\ -c_2 & c_2+c_3 & -c_3 & 0 & 0 & 0 \\ 0 & -c_3 & c_3+c_4 & -c_4 & 0 & 0 \\ 0 & 0 & -c_4 & c_4+c_5 & -c_5 & 0 \\ 0 & 0 & 0 & -c_5 & c_5+c_6 & -c_6 \\ 0 & 0 & 0 & 0 & -c_6 & c_6+c_7 \end{bmatrix} & \begin{array}{l} \text{mass 1} \\ \text{mass 2} \\ \text{mass 3} \\ \text{mass 4} \\ \text{mass 5} \\ \text{mass 6} \end{array}
\end{array}$$

- Removing spring 1 gives "line of springs with free - fixed ends"
- Removing springs 1 & 7: "line of springs with free - free ends"

(Free - fixed)

$$A = \begin{bmatrix} -1 & 1 & 0 & \cdots & \\ 0 & -1 & 1 & 0 & \cdots \\ & & \ddots & & \\ \cdots & 0 & -1 & 1 & \\ \cdots & 0 & 0 & -1 & \end{bmatrix}$$

Remove top row.

$$K = \begin{bmatrix} \boxed{c_2} & -c_2 & 0 & \cdots \\ -c_2 & c_2+c_3 & -c_3 & \cdots \\ 0 & -c_3 & \ddots & \ddots \\ \cdots & 0 & -c_6 & c_6+c_7 \end{bmatrix}$$

Only one element changes.

(Free - free)

$$A = \begin{bmatrix} -1 & 1 & 0 & \cdots & \\ 0 & -1 & 1 & 0 & \cdots \\ & & \ddots & & \\ \cdots & 0 & -1 & 1 & 0 \\ & \cdots & 0 & -1 & 1 \end{bmatrix}$$

Remove top & bottom rows.

$$K = \begin{bmatrix} \boxed{c_2} & -c_2 & 0 & \cdots \\ -c_2 & c_2+c_3 & -c_3 & \cdots \\ & \ddots & \ddots & \ddots \\ & \ddots & c_5+c_6 & -c_6 \\ & \cdots & -c_6 & \boxed{c_6} \end{bmatrix}$$

Only two elements change.

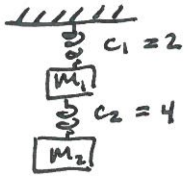
→ We will use these matrices later because A calculates "$-\frac{d}{dx}$" ← first derivative.

K calculates "$-\frac{d^2}{dx^2}$" ← second derivative.

Computational problems involve converting between displacement of masses (u) and force on masses (f) using the stiffness matrix

$$Ku = f$$

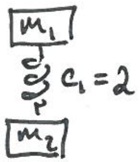
EX: Calculate the force required to hold the masses at displacements $u_1 = -1$ $\quad u_2 = 3$



$$\begin{bmatrix} 2+4 & -4 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$$

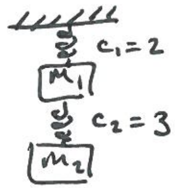
$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} 6 & -4 \\ -4 & 4 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -6-12 \\ 4+12 \end{bmatrix} = \begin{bmatrix} -18 \\ 16 \end{bmatrix}$$

EX: Calculate the force required to hold the masses at displacements $u_1 = -1$ $\quad u_2 = 4$



$$\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 4 \end{bmatrix} = \begin{bmatrix} -2-8 \\ 2+8 \end{bmatrix} = \begin{bmatrix} -10 \\ 10 \end{bmatrix}$$

EX: Calculate the displacements caused by the forces $f_1 = -1$ $\quad f_2 = 3$



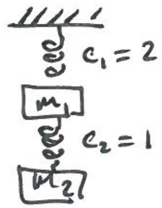
$$\begin{bmatrix} 2+3 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

2×2 inverse

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{1}{15-9} \begin{bmatrix} 3 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$= \frac{1}{6} \begin{bmatrix} -3+9 \\ -3+15 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

EX: Calculate the displacements caused by gravity if $m_1 = 2$ & $m_2 = 2$.



$$\begin{bmatrix} 2+1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = g \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

2×2 inverse

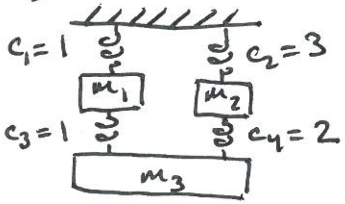
$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \frac{1}{3-1} \begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix} \cdot g \begin{bmatrix} 2 \\ 2 \end{bmatrix}$$

$$= \tfrac{1}{2} \cdot g \cdot \begin{bmatrix} 2+2 \\ 2+6 \end{bmatrix}$$

$$= g \cdot \begin{bmatrix} 2 \\ 4 \end{bmatrix}$$

(where g = gravitational constant)

EX: Calculate the displacements caused by the forces



$f_1=-6 \quad f_2=2 \quad f_3=9$

$$\begin{bmatrix}1+1&0&-1\\0&3+2&-2\\-1&-2&1+2\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=\begin{bmatrix}-6\\2\\9\end{bmatrix}$$

$$\begin{bmatrix}1+1&0&-1\\0&3+2&-2\\-1&-2&1+2\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\-1/2&0&1\end{bmatrix}\begin{bmatrix}2&0&-1\\0&5&-2\\0&-2&5/2\end{bmatrix} = \begin{bmatrix}1&0&0\\0&1&0\\-1/2&-2/5&1\end{bmatrix}\begin{bmatrix}2&0&-1\\0&5&-2\\0&0&17/10\end{bmatrix}$$

LU-Decomp.

$$\begin{bmatrix}1&0&0\\0&1&0\\-1/2&-2/5&1\end{bmatrix}\begin{bmatrix}2&0&-1\\0&5&-2\\0&0&17/10\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=\begin{bmatrix}-6\\2\\9\end{bmatrix}$$

$$\begin{bmatrix}1&0&0\\0&1&0\\-1/2&-2/5&1\end{bmatrix}\begin{bmatrix}a\\b\\c\end{bmatrix}=\begin{bmatrix}-6\\2\\9\end{bmatrix} \rightsquigarrow \begin{aligned}a&=-6\\b&=2\\c&=9-3+4/5\\&=34/5\end{aligned}$$

$$\begin{bmatrix}2&0&-1\\0&5&-2\\0&0&17/10\end{bmatrix}\begin{bmatrix}u_1\\u_2\\u_3\end{bmatrix}=\begin{bmatrix}-6\\2\\34/5\end{bmatrix} \rightsquigarrow \begin{aligned}u_3&=\frac{34/5}{17/10}=4\\u_2&=\frac{2+8}{5}=2\\u_1&=\frac{-6+4}{2}=-1\end{aligned}$$

EX: Calculate the displacements caused by the forces $f_1=9$, $f_2=-9$

$$\begin{bmatrix}3&-3\\-3&3\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}=\begin{bmatrix}9\\-9\end{bmatrix}$$

(Note: $\begin{bmatrix}3&-3\\-3&3\end{bmatrix}$ is not invertible because it has determinant $9-9=0$)

Try to solve with LU-decomposition

$$\begin{bmatrix}3&-3\\-3&3\end{bmatrix}=\begin{bmatrix}1&0\\-1&1\end{bmatrix}\begin{bmatrix}3&-3\\0&0\end{bmatrix}$$

(Divide by L) $\begin{bmatrix}1&0\\-1&1\end{bmatrix}\begin{bmatrix}a\\b\end{bmatrix}=\begin{bmatrix}9\\-9\end{bmatrix} \rightsquigarrow \begin{aligned}a&=9\\b&=0\end{aligned}$

(Divide by U) $\begin{bmatrix}3&-3\\0&0\end{bmatrix}\begin{bmatrix}u_1\\u_2\end{bmatrix}=\begin{bmatrix}9\\0\end{bmatrix} \rightsquigarrow \begin{aligned}u_2&=\text{free}\\u_1&=3+u_2\end{aligned}$

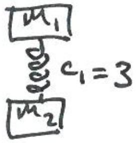
$$\begin{bmatrix}u_1\\u_2\end{bmatrix}=\begin{bmatrix}3\\0\end{bmatrix}+\begin{bmatrix}1\\1\end{bmatrix}u_2$$

distance between masses increases by 3

entire spring system can move up or down together.

(**Note:** It was important here that the two forces were equal in strength & opposite in direction.)

EX: Calculate the displacements caused by the forces $f_1 = 9$ $f_2 = -6$



$$\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 9 \\ -6 \end{bmatrix}$$

(LU-Decomp) $\begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix}$

(Divide by L) $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 9 \\ -6 \end{bmatrix} \leadsto \begin{array}{l} a = 9 \\ b = 3 \end{array}$

(Divide by U) $\boxed{\begin{bmatrix} 3 & -3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \end{bmatrix}} \leadsto 0 = 3$!!

No solution!

In this case because the masses are not attatched to something fixed, the entire system can shift together. The forces are not balanced — there is a net force of

$$f_1 + f_2 = 9 - 6 = 3$$

on the system. This will cause the entire system to shift downwards forever...

There is no equilibrium!